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Global well-posedness for the 2-D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity

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Abstract

We prove the global well-posedness for the 2-D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity.
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1. Introduction

We consider the Boussinesq model which is one of the most useful models in fluid and geophysical fluid dynamics and takes the form (see [20]):

$$\begin{cases} \partial_t u - \nabla\cdot(\nu\nabla u) + u\cdot\nabla u + \nabla p = \theta e_2, \quad e_2 = (0,1), \\ \nabla\cdot u = 0, \\ \partial_t\theta - \nabla\cdot(\kappa\nabla\theta) + u\cdot\nabla\theta = 0, \\ u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x). \end{cases} \tag{1.1}$$

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Here u and θ denote the velocity and the temperature of the fluid respectively. The viscosity ν and the thermal diffusivity κ depend on the temperature. Such dependence could be of great importance due to the large temperature contrast in certain applications. Throughout this paper, we assume that $\nu = \nu(\theta)$ and $\kappa = \kappa(\theta)$ for some smooth functions $\nu(\cdot)$ and $\kappa(\cdot)$ satisfy

$$C_0^{-1} \leqslant \nu(\theta) \leqslant C_0, \qquad C_0^{-1} \leqslant \kappa(\theta) \leqslant C_0, \quad \theta \in \mathbf{R}, \tag{1.2}$$

for some positive constants C_0.

In the case when ν and κ are positive constants, the global well-posedness of the 2-D Boussinesq system is classical. The local well-posedness can be easily established for the 3-D Boussinesq system by using the energy method. However, as in the incompressible Navier–Stokes equations, whether smooth solution blows up in finite time remains open. When ν and κ depend on the temperature, S.A. Lorca and J.L. Boldrini [17] proved the global existence of strong solution for small data. In [18], they obtained the global existence of weak solution and the local existence of strong solution for general data. Recently, M. Gunzburger, Y. Saka and X. Wang [8] also obtain the global well-posedness for the 3-D infinite Prandtl number model with temperature-dependent viscosity.

Recently, there are many works devoted to the study of the Boussinesq system with partial viscosity. That is, ν is a positive constant and $\kappa = 0$; or $\nu = 0$ and κ is a positive constant. D. Chae [4], and T.Y. Hou and C. Li [12] independently proved the global well-posedness of the 2-D Boussinesq system, see also [2,9] for the global well-posedness in the critical spaces and [15] for the case of bounded domain. R. Danchin and M. Paicu [6] proved the global existence of weak solution for L^2 data and the global well-posedness for small smooth data in 3-D case. T. Hmidi and F. Rousset [10,11] also proved the global well-posedness of the 3-D axisymmetric Boussinesq system without swirl.

In this paper, we are concerned with the case when ν and κ depend on the temperature. The goal is to prove the global well-posedness of the 2-D Boussinesq system. In this case, the energy estimates only yield that (u, θ) is bounded in $L^\infty(0, T; L^2(\mathbf{R}^2)) \cap L^2(0, T; H^1(\mathbf{R}^2))$. While, from the local well-posedness theory, we need to prove that $(\nabla u, \nabla\theta)$ is bounded in $L^2(0, T; L^\infty(\mathbf{R}^2))$ in order to extend the local solution to the global one, see Theorem 3.1. For this purpose, we first use the De-Giorgi method to prove θ is Hölder continuous. Next, we do the gradient estimate of θ. Since $u\theta$ is not bounded ($u \in L^2(0, T; BMO(\mathbf{R}^2))$ by Sobolev embedding), the Littlewood–Paley theory and the paraproduct technique play an important role in this step. With $\nabla\theta \in L^2(0, T; L^\infty(\mathbf{R}^2))$, it is easy to prove $u \in L^2(0, T; H^2(\mathbf{R}^2))$ by using the energy estimate. Hence, $\nabla u \in L^2(0, T; BMO(\mathbf{R}^2))$. So, there is still a small gap for the desired estimate of the velocity, since $L^\infty \subsetneq BMO$. However, this gap can be filled by combining the energy estimate with the Logarithmic Sobolev inequality.

Our main result is stated as follows.

Theorem 1.1. *Let $s > 2$, and $(u_0, \theta_0) \in H^s(\mathbf{R}^2)$. Then the Boussinesq system* (1.1) *has a unique global solution (u, θ) on $[0, +\infty)$ such that*

$$(u, \theta) \in C\big([0, +\infty); H^s\big(\mathbf{R}^2\big)\big) \cap L^2\big(0, T; H^{s+1}\big(\mathbf{R}^2\big)\big)$$

for any $T < +\infty$.

Remark 1.2. The same result can be easily generalized to the case of $\mathbf{T}^2$. In a separate paper, we will consider the initial boundary value problem in the bounded domain.

Remark 1.3. In the case when $\nu = \nu(\theta) > 0$ and $\kappa = 0$, the global existence of smooth solution for the 2-D Boussinesq system remains open. In the case when the initial temperature is small in some sense, H. Abidi [1] proved the global well-posedness of the 2-D Boussinesq system, see also [7] for a related result.

2. Preliminaries

Let us first recall some basic facts about the Littlewood–Paley theory, see [19] for more details. Let φ, χ be two functions in $C^\infty(\mathbf{R}^d)$ such that $\operatorname{supp}\hat{\varphi} \subset \{\frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$, $\operatorname{supp}\hat{\chi} \subset \{|\xi| \leqslant \frac{4}{3}\}$ and

$$\hat{\chi}(\xi) + \sum_{j\geqslant 0} \hat{\varphi}\big(2^{-j}\xi\big) = 1.$$

Then the Littlewood–Paley operators are defined by

$$\Delta_j f = \varphi_j * f = \int_{\mathbf{R}^d} \varphi_j(x-y) f(y)\,dy, \qquad \varphi_j(x) = 2^{jd}\varphi\big(2^j x\big), \quad j \geqslant 0,$$

$$S_j f = \chi_j * f = \sum_{k=-1}^{j-1} \Delta_k f, \qquad \Delta_{-1} f = \chi * f.$$

Some classical spaces can be characterized in terms of Δ_j. Let $s \in \mathbf{R}$. The Sobolev space $H^s(\mathbf{R}^d)$ is defined by

$$H^s\big(\mathbf{R}^d\big) \stackrel{\text{def}}{=} \Big\{u \in \mathcal{D}'\big(\mathbf{R}^d\big)\colon\ \|u\|_{H^s}^2 \stackrel{\text{def}}{=} \sum_{j\geqslant -1} 2^{2js}\|\Delta_j u\|_{L^2}^2 < \infty\Big\}.$$

We denote by $(u, v)_{H^s}$ the inner product in $H^s(\mathbf{R}^d)$. And for $s \in (0, 1)$, the Hölder space $C^s(\mathbf{R}^d)$ is defined by

$$C^s\big(\mathbf{R}^d\big) \stackrel{\text{def}}{=} \Big\{u \in \mathcal{D}'\big(\mathbf{R}^d\big)\colon\ \|u\|_{C^s} \stackrel{\text{def}}{=} \sup_{j\geqslant -1} 2^{js}\|\Delta_j u\|_{L^\infty}\Big\}.$$

We will use Bony's decomposition from [3]:

$$fg = T_f g + T_g f + R(f, g), \tag{2.1}$$

where

$$T_f g = \sum_{j\geqslant -1} S_{j-1} f \Delta_j g, \qquad R(f, g) = \sum_{|j-j'|\leqslant 1} \Delta_j f \Delta_{j'} g.$$

We also denote $T'_g f = T_g f + R(f, g)$.

Next we recall some lemmas which will be used throughout this paper.

Lemma 2.1. *(See [5].) Let $k \in \mathbf{N}$, $1 \leqslant p \leqslant q \leqslant \infty$. Then there exists a positive constant C independent of j such that*

$$\left\|\partial^\alpha \Delta_j f\right\|_{L^q} + \left\|\partial^\alpha S_j f\right\|_{L^q} \leqslant C 2^{j|\alpha| + dj(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p},$$
$$\|\Delta_j f\|_{L^p} \leqslant C 2^{-jk} \sup_{|\alpha|=k} \left\|\partial^\alpha \Delta_j f\right\|_{L^p}, \quad j \geqslant 0.$$

Lemma 2.2. *(See [13].) Let $s \geqslant 0$, $f, g \in H^s(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$. There holds*

$$\|fg\|_{H^s} \leqslant C\big(\|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}\big).$$

Lemma 2.3. *(See [13].) Let $s > 0$, $f \in H^{s-1}(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$, and $g \in H^s(\mathbf{R}^d) \cap W^{1,\infty}(\mathbf{R}^d)$. Then we have*

$$\big\|\big[\langle D\rangle^s, g\big] f\big\|_{L^2} \leqslant C\big(\|\nabla g\|_{L^\infty}\|f\|_{H^{s-1}} + \|g\|_{H^s}\|f\|_{L^\infty}\big),$$

where $\langle D\rangle^s$ is a Fourier multiplier operator with the symbol $(1+|\xi|^2)^{\frac{s}{2}}$.

Lemma 2.4. *(See [19].) Let $s > 0$, $f \in H^s(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$. Assume that $F(\cdot)$ is a smooth function on $\mathbf{R}$ with $F(0) = 0$. Then we have*

$$\big\|F(f)\big\|_{H^s} \leqslant C\big(1 + \|f\|_{L^\infty}\big)^{\lfloor s\rfloor + 1}\|f\|_{H^s},$$

where the constant C depends on $\sup_{k \leqslant \lfloor s\rfloor + 2,\, |t| \leqslant \|f\|_{L^\infty}} \|F^{(k)}(t)\|_{L^\infty}$.

Lemma 2.5 *(Logarithmic Sobolev inequality). Let $f \in H^s(\mathbf{R}^d)$ with $s > \frac{d}{2}$. Then there holds*

$$\|f\|_{L^\infty} \leqslant C\big(1 + \|f\|_{H^{\frac{d}{2}}}\big)\log^{\frac{1}{2}}\big(e + \|f\|_{H^s}\big).$$

Proof. One can refer to [14] for more general form. For the reader's convenience, here we give a proof. Using Littlewood–Paley decomposition, we decompose f into

$$f = \sum_{j=-1}^{N} \Delta_j f + \sum_{j \geqslant N+1} \Delta_j f,$$

from which and Lemma 2.1, we infer that

$$\begin{aligned}\|f\|_{L^\infty} &\leqslant C \sum_{j=-1}^{N} 2^{\frac{dj}{2}} \|\Delta_j f\|_{L^2} + C \sum_{j \geqslant N+1} 2^{\frac{dj}{2}} \|\Delta_j f\|_{L^2} \\ &\leqslant C(N+2)^{\frac{1}{2}} \|f\|_{H^{\frac{d}{2}}} + C 2^{-(s-\frac{d}{2})N} \|f\|_{H^s}.\end{aligned}$$

Taking $N \in \mathbf{N}$ such that

$$C2^{-(s-\frac{d}{2})N}\|f\|_{H^s} \sim 1,$$

the desired estimate follows easily. $\square$

3. Local well-posedness

This section is devoted to the proof of the local well-posedness of the system (1.1).

Theorem 3.1. *Let $s > 2$, and $(u_0, \theta_0) \in H^s(\mathbf{R}^2)$. Then there exist $T > 0$ and a unique solution (u, θ) on $[0, T)$ of the Boussinesq system* (1.1) *such that*

$$(u, \theta) \in C\big([0, T]; H^s\big(\mathbf{R}^2\big)\big) \cap L^2\big(0, T; H^{s+1}\big(\mathbf{R}^2\big)\big).$$

Furthermore, there holds

$$\begin{aligned}\big\|u(t)\big\|_{H^s}^2 + \big\|\theta(t)\big\|_{H^s}^2 + \int_0^t \big(\big\|\nabla u(\tau)\big\|_{H^s}^2 + \big\|\nabla\theta(\tau)\big\|_{H^s}^2\big)\, d\tau \\ \leqslant \big(\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2\big) \exp\left(\int_0^t G(\tau)\, d\tau\right),\end{aligned} \tag{3.1}$$

with $G(t)$ given by

$$G(t) = \mathcal{F}\big(\big\|\theta(t)\big\|_{L^\infty}\big)\big(1 + \big\|u(t)\big\|_{L^\infty}^2 + \big\|\nabla u(t)\big\|_{L^\infty}^2 + \big\|\nabla\theta(t)\big\|_{L^\infty}^2\big),$$

where $\mathcal{F}(\cdot)$ is a nondecreasing function on $\mathbf{R}^+$.

Proof. The proof is based on the energy method. We divide it into several steps.

Step 1. Construction of an approximate solution sequence.

We will use the Friedrich's method to construct the approximate solutions. Let us define the projector operator P_n by

$$P_n f(x) = \mathcal{F}^{-1}\big(1_{B_n}(\xi)\mathcal{F}f(\xi)\big)(x), \qquad \mathcal{F}f(\xi) = \int_{\mathbf{R}^2} f(x)e^{-ix\cdot\xi}\, dx,$$

where 1_{B_n} is a characteristic function on the ball B_n centered at the origin with radius n.

Next we introduce the following approximate system of (1.1):

$$\begin{cases} \partial_t u_n + P_n \mathrm{P}(P_n u_n \cdot \nabla P_n u_n) - P_n \mathrm{P}\nabla \cdot (\nu_n \nabla P_n u_n) = \mathrm{P}(P_n\theta_n e_2), \\ \partial_t \theta_n - P_n \nabla \cdot (\kappa_n \nabla P_n \theta_n) + P_n(P_n u_n \cdot \nabla P_n \theta_n) = 0, \\ u_n(0, x) = P_n u_0(x), \quad \theta_n(0, x) = P_n \theta_0(x). \end{cases} \tag{3.2}$$

Here $\nu_n = \nu(P_n\theta_n)$, $\kappa_n = \kappa(P_n\theta_n)$, and P denotes the Helmholtz projection operator onto divergence-free fields, which is given by

$$\mathrm{P} = (\delta_{ij} + R_i R_j)_{1\leqslant i,j\leqslant 2}$$

with Riesz transform R_i defined by $\mathcal{F}(R_i f)(\xi) = \frac{i\xi_i}{|\xi|}\mathcal{F}f(\xi)$, hence $P_n\mathrm{P} = \mathrm{P}P_n$.

Since all terms in (3.2) are bounded in $L^2(\mathbf{R}^2)$, the system (3.2) can be viewed as an ordinary differential system in $L^2(\mathbf{R}^2)$. Then the Cauchy–Lipschitz theorem ensures that the system (3.2) has a unique solution $(u_n, \theta_n) \in C([0, T_n]; L^2(\mathbf{R}^2))$ for some $T_n > 0$. Note that $P_n^2 = P_n$, $(P_n u_n, P_n\theta_n)$ is also a solution of (3.2). Thus the uniqueness of the solution implies that $(P_n u_n, P_n\theta_n) = (u_n, \theta_n)$. Hence, the approximate system (3.2) can be rewritten as

$$\begin{cases} \partial_t u_n + P_n\mathrm{P}(u_n\cdot\nabla u_n) - P_n\mathrm{P}\nabla\cdot(\nu_n\nabla u_n) = \mathrm{P}(\theta_n e_2), \\ \partial_t\theta_n - P_n\nabla\cdot(\kappa_n\nabla\theta_n) + P_n(u_n\cdot\nabla\theta_n) = 0, \\ u_n(0,x) = P_n u_0(x), \quad \theta_n(0,x) = P_n\theta_0(x), \end{cases} \tag{3.3}$$

with $\nu_n = \nu(\theta_n)$ and $\kappa_n = \kappa(\theta_n)$. On the other hand, the solution (u_n, θ_n) is in fact smooth due to $(P_n u_n, P_n\theta_n) = (u_n, \theta_n)$.

Step 2. Energy estimates.

Taking the $H^s(\mathbf{R}^2)$ inner product of the first equation of (3.3) with u_n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{H^s}^2 + (\nu_n\nabla u_n, \nabla u_n)_{H^s} = -(u_n\cdot\nabla u_n, u_n)_{H^s} + (\theta_n e_2, u_n)_{H^s}.$$

We get by Lemma 2.2 that

$$\begin{aligned} \left|(u_n\cdot\nabla u_n, u_n)_{H^s}\right| &= \left|(u_n\otimes u_n, \nabla u_n)_{H^s}\right| \leqslant \|u_n\otimes u_n\|_{H^s}\|\nabla u_n\|_{H^s} \\ &\leqslant C\|u_n\|_{L^\infty}\|u_n\|_{H^s}\|\nabla u_n\|_{H^s} \\ &\leqslant C\|u_n\|_{L^\infty}^2\|u_n\|_{H^s}^2 + \frac{C_0^{-1}}{4}\|\nabla u_n\|_{H^s}^2, \end{aligned}$$

and obviously,

$$\left|(\theta_n e_2, u_n)_{H^s}\right| \leqslant \|u_n\|_{H^s}\|\theta_n\|_{H^s} \leqslant \frac{1}{2}\|u_n\|_{H^s}^2 + \frac{1}{2}\|\theta_n\|_{H^s}^2.$$

For the second term of the left-hand side, we have

$$\begin{aligned} (\nu_n\nabla u_n, \nabla u_n)_{H^s} &= \big(\langle D\rangle^s(\nu_n\nabla u_n), \nabla\langle D\rangle^s u_n\big)_{L^2} \\ &= \big(\nu_n\nabla\langle D\rangle^s u_n, \nabla\langle D\rangle^s u_n\big)_{L^2} + \big(\big[\langle D\rangle^s, \nu_n\big]\nabla u_n, \nabla\langle D\rangle^s u_n\big)_{L^2}, \end{aligned}$$

which along with Lemmas 2.3 and 2.4 implies that

$$
\begin{aligned}
&(\nu_n \nabla u_n, \nabla u_n)_{H^s} \\
&\quad \geqslant C_0^{-1} \|\nabla u_n\|_{H^s}^2 - \mathcal{F}\big(\|\theta_n\|_{L^\infty}\big)\big(\|\nabla u_n\|_{L^\infty}\|\theta_n\|_{H^s} + \|\nabla\theta_n\|_{L^\infty}\|u_n\|_{H^s}\big)\|\nabla u_n\|_{H^s} \\
&\quad \geqslant \frac{C_0^{-1}}{2}\|\nabla u_n\|_{H^s}^2 - \mathcal{F}\big(\|\theta_n\|_{L^\infty}\big)\big(\|\nabla u_n\|_{L^\infty}^2\|\theta_n\|_{H^s}^2 + \|\nabla\theta_n\|_{L^\infty}^2\|u_n\|_{H^s}^2\big).
\end{aligned}
$$

Summing up all the above estimates yields that

$$
\begin{aligned}
&\frac{d}{dt}\|u_n\|_{H^s}^2 + \frac{C_0^{-1}}{4}\|\nabla u_n\|_{H^s}^2 \\
&\quad \leqslant \mathcal{F}\big(\|\theta_n\|_{L^\infty}\big)\big(1 + \|u_n\|_{L^\infty}^2 + \|\nabla u_n\|_{L^\infty}^2 + \|\nabla\theta_n\|_{L^\infty}^2\big)\big(\|u_n\|_{H^s}^2 + \|\theta_n\|_{H^s}^2\big).
\end{aligned}
$$

Similarly, it can be proved that

$$
\frac{d}{dt}\|\theta_n\|_{H^s}^2 + \frac{C_0^{-1}}{4}\|\nabla\theta_n\|_{H^s}^2 \leqslant \mathcal{F}\big(\|\theta_n\|_{L^\infty}\big)\big(1 + \|u_n\|_{L^\infty}^2 + \|\nabla\theta_n\|_{L^\infty}^2\big)\big(\|u_n\|_{H^s}^2 + \|\theta_n\|_{H^s}^2\big).
$$

Consequently, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\big(\|u_n\|_{H^s}^2 + \|\theta_n\|_{H^s}^2\big) + \frac{C_0^{-1}}{4}\big(\|\nabla u_n\|_{H^s}^2 + \|\nabla\theta_n\|_{H^s}^2\big) \\
&\quad \leqslant \mathcal{F}\big(\|\theta_n\|_{L^\infty}\big)\big(1 + \|u_n\|_{L^\infty}^2 + \|\nabla u_n\|_{L^\infty}^2 + \|\nabla\theta_n\|_{L^\infty}^2\big)\big(\|u_n\|_{H^s}^2 + \|\theta_n\|_{H^s}^2\big),
\end{aligned}
$$

from which and Gronwall's inequality, it follows that

$$
\begin{aligned}
E_n(t) &\stackrel{\text{def}}{=} \big\|u_n(t)\big\|_{H^s}^2 + \big\|\theta_n(t)\big\|_{H^s}^2 + \int_0^t \big(\big\|\nabla u_n(\tau)\big\|_{H^s}^2 + \big\|\nabla\theta_n(\tau)\big\|_{H^s}^2\big)\,d\tau \\
&\leqslant \big(\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2\big)\exp\left(\int_0^t G_n(\tau)\,d\tau\right),
\end{aligned}
\tag{3.4}
$$

with $G_n(t) = \mathcal{F}(\|\theta_n(t)\|_{L^\infty})(1 + \|u_n(t)\|_{L^\infty}^2 + \|\nabla u_n(t)\|_{L^\infty}^2 + \|\nabla\theta_n(t)\|_{L^\infty}^2)$.

Step 3. Uniform estimates and existence of the solution.

We denote T_n^* by the maximal existence time of the solution (u_n, θ_n) and define

$$
\widetilde{T}_n^* \stackrel{\text{def}}{=} \sup\big\{t \in \big[0, T_n^*\big):\ E_n(\tau) \leqslant 2\big(\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2\big) \text{ for } \tau \in [0, t]\big\}.
$$

From (3.4) and Sobolev embedding, we find that

$$
E_n(t) \leqslant \big(\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2\big)\exp\big(\mathcal{A}\big(\|u_0\|_{H^s} + \|\theta_0\|_{H^s}\big)t\big), \quad t \in \big[0, \widetilde{T}_n^*\big).
$$

Here $\mathcal{A}(\cdot)$ is some increasing function. Take T be small enough such that

$$
\exp\big(\mathcal{A}\big(\|u_0\|_{H^s} + \|\theta_0\|_{H^s}\big)T\big) \leqslant \frac{3}{2}.
$$

Now we can conclude that $\widetilde{T}_n^* \geqslant T$. Otherwise, we have

$$E_n(t) \leqslant \frac{3}{2}\big(\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2\big) \quad \text{for } t \in \big[0, \widetilde{T}_n^*\big],$$

which contradicts with the definition of $\widetilde{T}_n^*$. Thus, the approximate solution (u_n, θ_n) exists on $[0, T]$ and satisfies the following uniform estimate

$$E_n(t) \leqslant 2\big(\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2\big), \quad \text{for } t \in [0, T]. \tag{3.5}$$

On the other hand, it is easy to verify by using Eqs. (3.3) and (3.5) that $(\partial_t u_n, \partial_t \theta_n)$ is uniformly bounded in $L^2(0, T; H^{s-1}(\mathbf{R}^2))$. Thus, Lions–Aubin's compactness theorem ensures that there exist a subsequence $(u_{n_k}, \theta_{n_k})_k$ of $(u_n, \theta_n)_n$ and a function $(u, \theta) \in L^\infty(0, T; H^s(\mathbf{R}^2)) \cap L^2(0, T; H^{s+1}(\mathbf{R}^2))$ such that

$$\begin{aligned} u_{n_k} &\to u, \quad \text{in } L^2\big(0, T; H_{loc}^{s'+1}\big(\mathbf{R}^2\big)\big), \\ \theta_{n_k} &\to \theta, \quad \text{in } L^2\big(0, T; H_{loc}^{s'+1}\big(\mathbf{R}^2\big)\big), \end{aligned}$$

as $k \to +\infty$, for any $s' < s$. Then passing to limit in (3.3), it is easy to see that (u, θ) satisfies (1.1) in the weak sense and (u, θ) satisfies (3.1).

Step 4. Continuity in time of the solution.

Revisiting the proof of Step 2, one can in fact obtain better uniform estimates for (u_n, θ_n) (thus for (u, θ)):

$$\|u\|_{\widetilde{L}^\infty(0,T;H^s)} + \|\theta\|_{\widetilde{L}^\infty(0,T;H^s)} \leqslant C,$$

where $\|f\|_{\widetilde{L}^\infty(0,T;H^s)}^2 \stackrel{\text{def}}{=} \sum_{j \geqslant -1} 2^{2js} \|\Delta_j f\|_{L^\infty(0,T;L^2)}^2$. Then we can conclude $(u, \theta) \in C([0, T]; H^s(\mathbf{R}^2))$. Indeed, for any $\varepsilon > 0$, take N big enough such that

$$\sum_{j > N} 2^{2js} \|\Delta_j u\|_{L^\infty(0,T;L^2)}^2 \leqslant \frac{\varepsilon}{4}.$$

For any $t \in (0, T)$ and δ such that $t + \delta \in [0, T]$, we have

$$\begin{aligned} \big\|u(t+\delta) - u(t)\big\|_{H^s}^2 &\leqslant \sum_{j=-1}^{N} 2^{2js} \big\|\Delta_j u(t+\delta) - \Delta_j u(t)\big\|_{L^2}^2 + \frac{\varepsilon}{2} \\ &\leqslant \sum_{j=-1}^{N} 2^{2js} |\delta| \|\partial_t u\|_{L^2(0,T;L^2)}^2 + \frac{\varepsilon}{2} \\ &\leqslant 2N 2^{2sN} \|\partial_t u\|_{L^2(0,T;L^2)}^2 |\delta| + \frac{\varepsilon}{2} \leqslant \varepsilon, \end{aligned}$$

for $|\delta|$ small enough. Hence, $u(t)$ is continuous in $H^s(\mathbf{R}^2)$ at the time t.

Step 5. Uniqueness of the solution.

Let (u_1, θ_1) and (u_2, θ_2) be two solutions of (1.1) with the same initial data. We denote $\delta_u = u_1 - u_2$ and $\delta_\theta = \theta_1 - \theta_2$. Then $(\delta_u, \delta_\theta)$ satisfies

$$\begin{cases} \partial_t \delta_u - \nabla \cdot \big(\nu(\theta_1) \nabla \delta_u\big) + \nabla p = \delta_\theta e_2 - u_1 \cdot \nabla \delta_u - \delta_u \cdot \nabla u_2 + \nabla \cdot \big(\big(\nu(\theta_1) - \nu(\theta_2)\big) \nabla u_2\big), \\ \partial_t \delta_\theta - \nabla \cdot \big(\kappa(\theta_1) \nabla \delta_\theta\big) = -u_1 \cdot \nabla \delta_\theta - \delta_u \cdot \nabla \theta_2 + \nabla \cdot \big(\big(\kappa(\theta_1) - \kappa(\theta_2)\big) \nabla \theta_2\big). \end{cases}$$

Taking $L^2(\mathbf{R}^2)$ energy estimate, it is easy to show that

$$\frac{d}{dt}\big(\|\delta_u\|_{L^2}^2 + \|\delta_\theta\|_{L^2}^2\big) \leqslant C\big(\|\delta_u\|_{L^2}^2 + \|\delta_\theta\|_{L^2}^2\big),$$

which along with Gronwall's inequality gives $\delta_u = 0$ and $\delta_\theta = 0$.

The proof of Theorem 3.1 is completed. □

4. Hölder continuity for the temperature

We consider the temperature equation

$$\partial_t \theta - \nabla \cdot (\kappa \nabla \theta) + u \cdot \nabla \theta = 0, \qquad \theta(0, x) = \theta_0(x). \tag{4.1}$$

Our goal is to prove Hölder continuity of the temperature. More precisely,

Proposition 4.1. *Let $u \in L^4(0, T; L^4(\mathbf{R}^2))$ be a divergence-free field. Assume that $\theta \in L^\infty(0, T; L^2(\mathbf{R}^2)) \cap L^2(0, T; H^1(\mathbf{R}^2))$ is a weak solution of* (4.1). *Then there exists $\alpha > 0$ such that $\theta \in C^\alpha((0, T] \times \mathbf{R}^2)$ with*

$$\|\theta\|_{L^\infty(\delta, T; C^\alpha)} \leqslant C\big(\delta, \|u\|_{L^4(0,T;L^4)}, \|\theta_0\|_{L^2}\big)$$

for any $\delta \in (0, T)$.

Remark 4.2. If $u \in L^p(0, T; L^p(\mathbf{R}^2))$ for some $p > 4$, the Hölder continuity of θ is a classical result, see [16]. In the present case, $\operatorname{div} u = 0$ plays an important role.

Let us first introduce some notations. Given $Y = (s, y) \in \mathbf{R}^+ \times \mathbf{R}^d$, $R > 0$, $\tau > 0$, $k \in \mathbf{R}$, we denote

$$Q(Y, R, \tau) \stackrel{\text{def}}{=} \big\{X = (t, x)\colon |x - y| < R,\ s - \tau R^2 < t < s\big\}, \qquad Q(Y, R) = Q(Y, R, 1),$$
$$A^{\pm}_{k,R,\tau}(Y) \stackrel{\text{def}}{=} \big\{X \in Q(Y, R, \tau)\colon \pm(\theta - k)(X) > 0\big\},$$
$$A^{\pm}_{k,R}(Y) \stackrel{\text{def}}{=} \big\{X \in Q(Y, R)\colon \pm(\theta - k)(X) > 0\big\}.$$

We denote $(\theta - k)_+ = \max(\theta - k, 0)$ and $(\theta - k)_- = \max(-\theta + k, 0)$. We say $\theta \in PDG^+(Q(Y, R))$ if $\theta \in V \stackrel{\text{def}}{=} L^\infty(s - R^2, s; L^2(B_R)) \cap L^2(Q(Y, R))$, and there exist $0 < \varepsilon \leqslant \frac{2}{d+2}$, $C_0 > 0$

and an increasing function $\chi(\cdot)$ such that

$$\sup_{s-\sigma^2R^2<t<s}\int_{B_{\sigma R}}\left|(\theta-k)_+(t,x)\right|^2dx+\int_{Q(\sigma R)}\left|\nabla(\theta-k)_+\right|^2dx\,dt$$
$$\leqslant\frac{C_0}{(1-\sigma)^2R^2}\int_{Q(R)}\left|(\theta-k)_+\right|^2dx\,dt+C_0\bigl(\chi(R)^2+R^{-(d+2)\varepsilon}k^2\bigr)\left|A^+_{k,R}\right|^{1+\varepsilon-\frac{2}{d+2}}$$

for any $\sigma\in(0,1)$, $k>0$, and $Q(R)=Q(Y,R)$. We say that $u\in PDG^-(Q(Y,R))$ if $\theta\in V$, and there exist $0<\varepsilon\leqslant\frac{2}{d+2}$, $C_0>0$ and an increasing function $\chi(\cdot)$ such that

$$\sup_{s-\sigma^2R^2<t<s}\int_{B_{\sigma R}}\left|(\theta-k)_-(t,x)\right|^2dx+\int_{Q(\sigma R)}\left|\nabla(\theta-k)_-\right|^2dx\,dt$$
$$\leqslant\frac{C_0}{(1-\sigma)^2R^2}\int_{Q(R)}\left|(\theta-k)_-\right|^2dx\,dt+C_0\bigl(\chi(R)^2+R^{-(d+2)\varepsilon}k^2\bigr)\left|A^-_{k,R}\right|^{1+\varepsilon-\frac{2}{d+2}}$$

for any $\sigma\in(0,1)$, $k>0$; and also

$$\sup_{s_0<t<s}\int_{B_{\sigma R}}\left|(\theta-k)_-(t,x)\right|^2dx$$
$$\leqslant\int_{B_{\sigma R}}\left|(\theta-k)_-(s_0,x)\right|^2dx+\frac{C_0}{(1-\sigma)^2R^2}\int_{Q(R,\tau)}\left|(\theta-k)_-\right|^2dx\,dt$$
$$+C_0\bigl(\chi(R)^2+R^{-(d+2)\varepsilon}k^2\bigr)\left|A^-_{k,R,\tau}\right|^{1+\varepsilon-\frac{2}{d+2}}$$

for any $\sigma\in(0,1)$, $k>0$, $s_0>s-\tau R^2$, and $Q(R,\tau)=Q(Y,R,\tau)$.

Using the De-Giorgi method, the Hölder continuity can be deduced by showing that θ belongs to the parabolic De-Giorgi class (PDG). That is,

Lemma 4.1. *(See [16].) Assume that θ is a bounded function in $Q(Y,R)$, and $\theta\in PDG^\pm(Q(Y,R))$. Then there exists $\alpha>0$ depending on ε, C_0 such that $\theta\in C^\alpha(Q(Y,\frac{R}{2}))$ and*

$$\|\theta\|_{C^\alpha(Q(Y,\frac{R}{2}))}\leqslant C\bigl(\varepsilon,C_0,\|\theta\|_{L^\infty(Q(Y,R))}\bigr).$$

Lemma 4.2. *Under the same assumptions as Proposition* 4.1*, the solution θ is bounded in $(\delta,T)\times\mathbf{R}^2$ for any $\delta>0$ and*

$$\|\theta\|_{L^\infty((\delta,T)\times\mathbf{R}^2)}\leqslant C\bigl(\delta,\|\theta_0\|_{L^2}\bigr).$$

Proof. We set

$$C_k=M\bigl(1-2^{-k}\bigr),\qquad \theta_k=(\theta-C_k)_+,$$

where M will be chosen later. Multiplying (4.1) by θ_k, and integrating in $\mathbf{R}^2$, we obtain

$$\frac{1}{2}\partial_t \int_{\mathbf{R}^2} \theta_k^2\,dx + \int_{\mathbf{R}^2} \kappa|\nabla\theta_k|^2\,dx = -\frac{1}{2}\int_{\mathbf{R}^2} u\cdot\nabla\theta_k^2 = 0. \tag{4.2}$$

For any fixed $T > \delta > 0$, we introduce $T_k = \delta(1-2^{-k})$ and the level set of energy

$$U_k = \sup_{T_k \leqslant t \leqslant T} \int_{\mathbf{R}^2} \theta_k^2\,dx + 2\int_{T_k}^{T}\int_{\mathbf{R}^2} \kappa|\nabla\theta_k|^2\,dx\,dt.$$

Integrating (4.2) in time between $s \in (T_{k-1}, T_k)$ and $t > T_k$, we obtain

$$U_k \leqslant \int_{\mathbf{R}^2} \theta_k^2(s,x)\,dx.$$

Taking the mean value in s on $[T_{k-1}, T_k]$, we find

$$U_k \leqslant \frac{2^k}{\delta}\int_{T_{k-1}}^{T}\int_{\mathbf{R}^2} \theta_k^2\,dx\,dt. \tag{4.3}$$

Next, we want to control the right-hand side by U_{k-1} in a nonlinear way. By the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}\int_{T_{k-1}}^{T} \left\|\theta_{k-1}(t)\right\|_{L^4}^4\,dt &\leqslant \int_{T_{k-1}}^{T} \left\|\theta_{k-1}(t)\right\|_{L^2}^2 \left\|\nabla\theta_{k-1}(t)\right\|_{L^2}^2\,dt \\ &\leqslant CU_{k-1}^2.\end{aligned}$$

On the other hand, if $\theta_k > 0$ then

$$\theta_{k-1} = \theta - C_{k-1} = \theta - C_k + C_k - C_{k-1} \geqslant 2^{-k}M,$$

which means that

$$1_{\{\theta_k>0\}} \leqslant \frac{2^k\theta_{k-1}}{M},$$

from which and (4.3), it follows that

$$
\begin{aligned}
U_k &\leqslant \frac{2^k}{\delta} \int_{T_{k-1}}^{T} \int_{\mathbf{R}^2} \theta_{k-1}^2 1_{\{\theta_k>0\}}\, dx\, dt \\
&\leqslant \frac{2^{3k}}{M^2\delta} \int_{T_{k-1}}^{T} \int_{\mathbf{R}^2} \theta_{k-1}^4\, dx\, dt \\
&\leqslant C \frac{2^{3k}}{M^2\delta} U_{k-1}^2,
\end{aligned}
$$

from which, it is easy to deduce that $U_k \to 0$ as $k \to \infty$, if we take M big enough depending only on δ and $\|\theta_0\|_{L^2}$. Thus $\theta \leqslant M$, and similarly $-\theta \leqslant M$. □

Now we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Let $\eta(t,x)$ be a smooth cut-off function supported in $(0,T] \times \mathbf{R}^2$. We denote $\theta_k = (\theta - k)_+$ for $k \in \mathbf{R}$. Multiplying (4.1) by $\eta^2\theta_k$, and then integrating resulting equation on $[t_1, t] \times \mathbf{R}^2$ with $0 < t_1 < t \leqslant t_2 \leqslant T$, we obtain

$$
\int_{t_1}^{t} \int_{\mathbf{R}^2} \partial_t\theta\theta_k\eta^2\, dx\, dt - \int_{t_1}^{t} \int_{\mathbf{R}^2} \nabla\cdot(\kappa\nabla\theta)\theta_k\eta^2\, dx\, dt = -\int_{t_1}^{t} \int_{\mathbf{R}^2} u\cdot\nabla\theta\theta_k\eta^2\, dx\, dt.
$$

By integrating by parts, we get

$$
\int_{t_1}^{t} \int_{\mathbf{R}^2} \partial_t\theta\theta_k\eta^2\, dx\, dt = \frac{1}{2}\int_{\mathbf{R}^2} \theta_k^2\eta^2(t_2,x)\, dx - \frac{1}{2}\int_{\mathbf{R}^2} \theta_k^2\eta^2(t_1,x)\, dx - \int_{t_1}^{t} \int_{\mathbf{R}^2} \theta_k^2\partial_t\eta\eta\, dx\, dt,
$$

and

$$
-\int_{t_1}^{t} \int_{\mathbf{R}^2} \nabla\cdot(\kappa\nabla\theta)\theta_k\eta^2\, dx\, dt = \int_{t_1}^{t} \int_{\mathbf{R}^2} \kappa\left|\nabla(\theta_k\eta)\right|^2 dx\, dt - \int_{t_1}^{t} \int_{\mathbf{R}^2} \kappa\theta_k^2|\nabla\eta|^2\, dx\, dt.
$$

Due to $\nabla\cdot u = 0$, we get by integration by parts that

$$
-\int_{t_1}^{t} \int_{\mathbf{R}^2} u\cdot\nabla\theta\theta_k\eta^2\, dx\, dt = \int_{t_1}^{t} \int_{\mathbf{R}^2} \theta_k^2 u\cdot\nabla\eta\eta\, dx\, dt.
$$

Summing up the above identities, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^2} \theta_k^2\eta^2(t,x)\, dx + \int_{t_1}^{t} \int_{\mathbf{R}^2} \kappa\left|\nabla(\theta_k\eta)\right|^2 dx\, dt \\
&\quad = \frac{1}{2}\int_{\mathbf{R}^2} \theta_k^2\eta^2(t_1,x)\, dx + \int_{t_1}^{t} \int_{\mathbf{R}^2} \kappa\theta_k^2\left(|\nabla\eta|^2 + \partial_t\eta\eta\right) dx\, dt + \int_{t_1}^{t} \int_{\mathbf{R}^2} \theta_k^2 u\cdot\nabla\eta\eta\, dx\, dt.
\end{aligned}
$$

By Hölder's inequality and Gagliardo–Nirenberg's inequality, we have

$$\begin{aligned}
&\left|\int_{t_1}^{t}\int_{\mathbf{R}^2}\theta_k^2 u\cdot\nabla\eta\eta\,dx\,dt\right| \\
&\leqslant \|u\|_{L^4(t_1,t_2;L^4(\mathbf{R}^2))}\left(\int_{t_1}^{t_2}\int_{\mathbf{R}^2}|\theta_k|^{\frac{8}{3}}|\nabla\eta|^{\frac{4}{3}}|\eta|^{\frac{4}{3}}\,dx\,dt\right)^{\frac{3}{4}} \\
&\leqslant C\|\eta\theta_k\|_{L^4((t_1,t_2)\times\mathbf{R}^2)}\|\nabla\eta\theta_k\|_{L^2((t_1,t_2)\times\mathbf{R}^2)} \\
&\leqslant C\|\eta\theta_k\|^{\frac{1}{2}}_{L^\infty((t_1,t_2);L^2(\mathbf{R}^2))}\big\|\nabla(\eta\theta_k)\big\|^{\frac{1}{2}}_{L^2((t_1,t_2)\times\mathbf{R}^2)}\|\nabla\eta\theta_k\|_{L^2((t_1,t_2)\times\mathbf{R}^2)} \\
&\leqslant \frac{1}{4}\|\eta\theta_k\|^2_{L^\infty((t_1,t_2);L^2(\mathbf{R}^2))}+\frac{C_0^{-1}}{2}\big\|\nabla(\eta\theta_k)\big\|^2_{L^2((t_1,t_2)\times\mathbf{R}^2)}+C\|\nabla\eta\theta_k\|^2_{L^2((t_1,t_2)\times\mathbf{R}^2)}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\sup_{t_1\leqslant t\leqslant t_2}\int_{\mathbf{R}^2}\theta_k^2\eta^2(t,x)\,dx+2C_0^{-1}\int_{t_1}^{t_2}\int_{\mathbf{R}^2}\big|\nabla(\theta_k\eta)\big|^2\,dx\,dt \\
&\quad\leqslant 2\int_{\mathbf{R}^2}\theta_k^2\eta^2(t_1,x)\,dx+C\int_{t_1}^{t_2}\int_{\mathbf{R}^2}\theta_k^2\big(|\nabla\eta|^2+|\partial_t\eta|\big)\,dx\,dt,
\end{aligned}$$

from which, it is easy to see that $\theta\in PDG^{\pm}$. Then Proposition 4.1 can be deduced from Lemmas 4.1 and 4.2. □

5. Gradient estimate for the temperature

In this section, we present the gradient estimate of the temperature by using the Littlewood–Paley analysis.

Proposition 5.1. *Let* $(u,\theta)\in C([0,T];H^s(\mathbf{R}^2))\cap L^2(0,T;H^{s+1}(\mathbf{R}^2))$ *be a solution of* (1.1). *Then there exists* $\delta>0$ *depending only on* $\|\theta\|_{L^\infty(0,T;C^\alpha(\mathbf{R}^2))}$ *such that*

$$\int_t^{\min(t+\delta,T)}\big\|\nabla\theta(\tau)\big\|^2_{L^\infty}\,d\tau\leqslant C\big(\big\|\theta(t)\big\|_{H^{1+\alpha}}+\|u\|_{L^2(0,T;H^1)}\|\theta\|_{L^\infty(0,T;C^\alpha)}\big)^2$$

for any $t\in[0,T)$.

Remark 5.2. If $u\in L^p((0,T)\times\mathbf{R}^2)$ for some $p>4$, then $\nabla\theta\in C^\alpha((0,T)\times\mathbf{R}^2)$ for some $\alpha>0$ by Theorem 4.8 in [16]. However, we only have $u\in L^4((0,T)\times\mathbf{R}^2)$ by Sobolev inequality, since $u\in L^\infty(0,T;L^2(\mathbf{R}^2))\cap L^2(0,T;H^1(\mathbf{R}^2))$ by the L^2 energy estimate. On the other hand, one may write $u\cdot\nabla\theta=\nabla\cdot(u\theta)$, thus $f=u\theta$ can be viewed as an external force. By Theorem 4.8

in [16] again, if $f \in C^\alpha((0,T)\times \mathbf{R}^2)$, then $\nabla\theta \in C^\alpha((0,T)\times \mathbf{R}^2)$. However, f is not bounded since $u \in L^2(0,T; BMO(\mathbf{R}^2))$ by Sobolev embedding.

Proof. We apply Δ_j to both sides of the temperature equation to obtain

$$\partial_t \Delta_j\theta - \nabla\cdot(\kappa\nabla\Delta_j\theta) = -\Delta_j(u\cdot\nabla\theta) + \nabla\cdot[\Delta_j,\kappa]\nabla\theta.$$

Multiplying it by $\Delta_j\theta$, and then integrating on $\mathbf{R}^2$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta\|_{L^2}^2 + C_0^{-1}2^{2j}\|\Delta_j\theta\|_{L^2}^2 \\
&\quad\leqslant -\int_{\mathbf{R}^2}[\Delta_j,u]\cdot\nabla\theta\,\Delta_j\theta\,dx + \int_{\mathbf{R}^2}\nabla\cdot[\Delta_j,\kappa]\nabla\theta\,\Delta_j\theta\,dx \\
&\quad\leqslant \big(\big\|[\Delta_j,u]\cdot\nabla\theta\big\|_{L^2} + C2^j\big\|[\Delta_j,\kappa]\nabla\theta\big\|_{L^2}\big)\|\Delta_j\theta\|_{L^2}.
\end{aligned}$$

Here we used the fact that

$$\int_{\mathbf{R}^2} u\cdot\nabla\Delta_j\theta\,\Delta_j\theta\,dx = -\frac{1}{2}\int_{\mathbf{R}^2}\nabla\cdot u|\Delta_j\theta|^2\,dx = 0.$$

Then it follows that

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^2} + C_0^{-1}2^{2j}\|\Delta_j\theta\|_{L^2} \leqslant \big\|[\Delta_j,u]\cdot\nabla\theta\big\|_{L^2} + C2^j\big\|[\Delta_j,\kappa]\nabla\theta\big\|_{L^2},$$

which implies that for any $t' \in [t,T]$,

$$\begin{aligned}
\big\|\Delta_j\theta(t')\big\|_{L^2} &\leqslant e^{-c(t'-t)2^{2j}}\big\|\Delta_j\theta(t)\big\|_{L^2} + 2\int_t^{t'} e^{-c(t'-\tau)2^{2j}}\big\|[\Delta_j,u]\cdot\nabla\theta\big\|_{L^2}\,d\tau \\
&\quad + C2^j\int_t^{t'} e^{-c(t'-\tau)2^{2j}}\big\|[\Delta_j,\kappa]\nabla\theta\big\|_{L^2}\,d\tau, \qquad c = C_0^{-1}.
\end{aligned}$$

Taking L^2 norm with respect to time on $[t,t+\delta]$ to get

$$\begin{aligned}
\big\|\Delta_j\theta(t')\big\|_{L^2(t,t+\delta;L^2)} &\leqslant C2^{-j}\big\|\Delta_j\theta(t)\big\|_{L^2} + C2^{-2j}\big\|[\Delta_j,u]\cdot\nabla\theta\big\|_{L^2(t,t+\delta;L^2)} \\
&\quad + C\delta^{\frac{\alpha}{2}}2^{-(1-\alpha)j}\big\|[\Delta_j,\kappa]\nabla\theta\big\|_{L^2(t,t+\delta;L^2)},
\end{aligned}$$

which together with Lemma A.1 yields that

$$\|\theta\|_{\widetilde{L}^2(t,t+\delta;B^{2+\alpha}_{2,\infty})} \leqslant C\big\|\theta(t)\big\|_{H^{1+\alpha}} + C\|\theta\|_{L^\infty(t,t+\delta;C^\alpha)}\big(\|u\|_{L^2(t,t+\delta;H^1)} + \delta^{\frac{\alpha}{2}}\|\theta\|_{\widetilde{L}^2(t,t+\delta;B^{2+\alpha}_{2,\infty})}\big),$$

where

$$\|\theta\|_{\widetilde{L}^2(t,t+\delta;B^s_{2,\infty})} \stackrel{\text{def}}{=} \sup_{j\geqslant -1} 2^{js}\|\Delta_j\theta\|_{L^2(t,t+\delta;L^2)}.$$

Taking δ small enough such that $C\|\theta\|_{L^\infty(0,T;C^\alpha)}\delta^{\frac{\alpha}{2}} \leqslant \frac{1}{2}$, we infer that

$$\|\theta\|_{\widetilde{L}^2(t,t+\delta;B^{2+\alpha}_{2,\infty})} \leqslant C\|\theta\|_{H^{1+\alpha}} + C\|\theta\|_{L^\infty(t,t+\delta;C^\alpha)}\|u\|_{L^2(t,t+\delta;H^1)},$$

and by Lemma 2.1

$$\begin{aligned}
\int_t^{t+\delta} \big\|\nabla\theta(\tau)\big\|^2_{L^\infty}\,d\tau &\leqslant \sum_{j\geqslant -1}\int_t^{t+\delta} \big\|\nabla\Delta_j\theta(\tau)\big\|^2_{L^\infty}\,d\tau \\
&\leqslant C\sum_{j\geqslant -1}\int_t^{t+\delta} 2^{4j}\big\|\Delta_j\theta(\tau)\big\|^2_{L^2}\,d\tau \\
&\leqslant C\sum_{j\geqslant -1} 2^{-2j\alpha}\|\theta\|^2_{\widetilde{L}^2(t,t+\delta;B^{2+\alpha}_{2,\infty})} \leqslant C\|\theta\|^2_{\widetilde{L}^2(t,t+\delta;B^{2+\alpha}_{2,\infty})},
\end{aligned}$$

which gives the desired estimate. □

6. Proof of Theorem 1.1

In this section, we prove the global well-posedness of the Boussinesq system (1.1). First of all, Theorem 3.1 provides us a local strong solution $(u,\theta) \in C([0,T];H^s(\mathbf{R}^2)) \cap L^2([0,T];H^{s+1}(\mathbf{R}^2))$. Let T^* be the maximal existence time of the solution. It suffices to prove $T^* = +\infty$. We will argue by contradiction argument. Hence, assume $T^* < +\infty$ in what follows.

Taking L^2 energy estimate to the system (1.1), it is easy to find that

$$\big\|(u,\theta)\big\|_{L^\infty(0,T^*;L^2)} + \big\|(\nabla u,\nabla\theta)\big\|_{L^2(0,T^*;L^2)} \leqslant \big\|(u_0,\theta_0)\big\|_{L^2}\exp\big(CT^*\big),$$

which implies that u is bounded in $L^4(0,T^*;L^4(\mathbf{R}^2))$ by Gagliardo–Nirenberg inequality. Then Proposition 4.1 ensures that the temperature θ is bounded in $L^\infty(0,T^*;C^\alpha(\mathbf{R}^2))$ for some $\alpha > 0$. Hence, we get by Proposition 5.1 that

$$\int_{T^*-\delta}^{T^*} \big\|\nabla\theta(t)\big\|^2_{L^\infty}\,dt \leqslant C\big(\big\|\theta\big(T^*-\delta\big)\big\|_{H^{1+\alpha}} + \|u\|_{L^2(0,T^*;H^1)}\|\theta\|_{L^\infty(0,T^*;C^\alpha)}\big)^2,$$

which implies that $\nabla\theta$ is bounded in $L^2(0,T^*;L^\infty(\mathbf{R}^2))$.

Next, take L^2 inner product of the velocity equation with Δu to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\int_{\mathbf{R}^2}\nu(\theta)|\Delta u|^2\,dx=(\theta e_2,\Delta u)+(-u\cdot\nabla u,\Delta u)+\big(\nabla\nu(\theta)\cdot\nabla u,\Delta u\big)$$
$$\leqslant\big(\|\theta\|_{L^2}+\|u\cdot\nabla u\|_{L^2}+\big\|\nabla\nu(\theta)\cdot\nabla u\big\|_{L^2}\big)\|\Delta u\|_{L^2}.$$

By Gagliardo–Nirenberg's inequality, we have

$$\|u\cdot\nabla u\|_{L^2}\leqslant C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}},$$

and by Hölder's inequality,

$$\big\|\nabla\nu(\theta)\cdot\nabla u\big\|_{L^2}\leqslant C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{L^2}.$$

Then we get by Young's inequality that

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2+\frac{C_0^{-1}}{2}\|\Delta u\|_{L^2}^2\leqslant C\big(\|\theta\|_{L^2}^2+\|u\|_{L^2}\|\nabla u\|_{L^2}^4+\|\nabla\theta\|_{L^\infty}^2\|\nabla u\|_{L^2}^2\big),$$

which along with Gronwall's inequality ensures that u is bounded in $L^2(0,T^*;H^2(\mathbf{R}^2))$.

On the other hand, by Theorem 3.1, (u,θ) satisfies the following energy estimate:

$$\big\|u(t)\big\|_{H^s}^2+\big\|\theta(t)\big\|_{H^s}^2\leqslant\big(\|u_0\|_{H^s}^2+\|\theta_0\|_{H^s}^2\big)\exp\left(\int_0^t G(\tau)\,d\tau\right).$$

Now by the above arguments and Lemma 2.5, we know that

$$\int_0^t G(\tau)\,d\tau\leqslant C+\int_0^t\big\|\nabla u(\tau)\big\|_{L^\infty}^2\,d\tau$$
$$\leqslant C+C\int_0^t\big(1+\big\|u(\tau)\big\|_{H^2}^2\big)\log\big(e+\big\|u(\tau)\big\|_{H^s}^2\big)\,d\tau,$$

for any $t\in[0,T^*]$. Thus, we have

$$\big\|u(t)\big\|_{H^s}^2+\big\|\theta(t)\big\|_{H^s}^2$$
$$\leqslant\big(\|u_0\|_{H^s}^2+\|\theta_0\|_{H^s}^2\big)\exp\left(C+C\int_0^t\big(1+\big\|u(\tau)\big\|_{H^2}^2\big)\log\big(e+\big\|u(\tau)\big\|_{H^s}^2\big)\,d\tau\right),$$

which implies that

$$\log\big(e+\|u(t)\|_{H^s}^2+\|\theta(t)\|_{H^s}^2\big)\leqslant\log\big(e+\|u_0\|_{H^s}^2+\|\theta_0\|_{H^s}^2\big)+C+C\int_0^t\big(1+\|u(\tau)\|_{H^2}^2\big)\log\big(e+\|u(\tau)\|_{H^s}^2+\|\theta(\tau)\|_{H^s}^2\big)\,d\tau.$$

Since u is bounded in $L^2(0,T^*;H^2(\mathbf{R}^2))$, Gronwall's inequality ensures that (u,θ) is also bounded in $L^\infty(0,T^*;H^s(\mathbf{R}^2))$. Thus, the solution can be extended after $t=T^*$, which contradicts with the definition of T^*. Hence, $T^*=+\infty$.

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Appendix A

Lemma A.1. *Let $\alpha\in(0,1)$. For any $j\geqslant-1$, there hold*

$$\big\|[\Delta_j,u]\cdot\nabla\theta\big\|_{L^2(0,T;L^2)}\leqslant C2^{-j\alpha}\|\theta\|_{L^\infty(0,T;C^\alpha)}\|u\|_{L^2(0,T;H^1)},$$
$$\big\|[\Delta_j,\kappa]\nabla\theta\big\|_{L^2(0,T;L^2)}\leqslant C2^{-j(1+2\alpha)}\|\theta\|_{L^\infty(0,T;C^\alpha)}\|\theta\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})},$$

where C depends on $\|\theta\|_{L^\infty(0,T;L^\infty)}$ in the second inequality.

Proof. We only present the proof of the second inequality, the proof of the first inequality is similar. Let $\tilde\kappa=\kappa(\theta)-\kappa(0)$. Then $[\Delta_j,\kappa]\nabla\theta=[\Delta_j,\tilde\kappa]\nabla\theta$.

Use Bony's decomposition (2.1) to decompose

$$\Delta_j(\tilde\kappa\nabla\theta)=\Delta_j(T_{\tilde\kappa}\nabla\theta)+\Delta_j(T_{\nabla\theta}\tilde\kappa)+\Delta_jR(\tilde\kappa,\nabla\theta),$$
$$\tilde\kappa\Delta_j\nabla\theta=T_{\tilde\kappa}\Delta_j\nabla\theta+T'_{\Delta_j\nabla\theta}\tilde\kappa.$$

Then we have

$$\begin{aligned}[\Delta_j,\tilde\kappa]\nabla\theta&=\Delta_j(\tilde\kappa\nabla\theta)-\tilde\kappa\Delta_j\nabla\theta\\&=[\Delta_j,T_{\tilde\kappa}]\nabla\theta+\Delta_j(T_{\nabla\theta}\tilde\kappa)+\Delta_jR(\tilde\kappa,\nabla\theta)-T'_{\Delta_j\nabla\theta}\tilde\kappa.\end{aligned}$$

Notice that

$$\Delta_j(T_{\nabla\theta}\tilde\kappa)=\sum_{|j-j'|\leqslant4}\Delta_j(S_{j'-1}\nabla\theta\Delta_{j'}\tilde\kappa).$$

This gives by Lemma 2.1 that

$$\begin{aligned}\big\|\Delta_j(T_{\nabla\theta}\tilde{\kappa})\big\|_{L^2(0,T;L^2)} &\leqslant C\sum_{|j-j'|\leqslant 4}\|S_{j'-1}\nabla\theta\|_{L^\infty(0,T;L^\infty)}\|\Delta_{j'}\tilde{\kappa}\|_{L^2(0,T;L^2)}\\ &\leqslant C2^{-j(1+2\alpha)}\|\theta\|_{L^\infty(0,T;C^\alpha)}\|\tilde{\kappa}\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})}.\end{aligned}$$

Here we used

$$\|S_{j'-1}\nabla\theta\|_{L^\infty(0,T;L^\infty)}\leqslant C\sum_{k\leqslant j'-2}2^k\|\Delta_k\theta\|_{L^\infty(0,T;L^\infty)}\leqslant C2^{j'(1-\alpha)}\|\theta\|_{L^\infty(0,T;C^\alpha)}.$$

Using the fact that

$$\Delta_j R(\tilde{\kappa},\nabla\theta)=\sum_{j',j''\geqslant j-3;\,|j'-j''|\leqslant 1}\Delta_j(\Delta_{j'}\tilde{\kappa}\Delta_{j''}\nabla\theta),$$

it follows from Lemma 2.1 that

$$\begin{aligned}\big\|\Delta_j R(\tilde{\kappa},\nabla\theta)\big\|_{L^2(0,T;L^2)} &\leqslant \sum_{j',j''\geqslant j-3;\,|j'-j''|\leqslant 1}2^{j''}\|\Delta_{j'}\tilde{\kappa}\|_{L^2(0,T;L^2)}\|\Delta_{j''}\theta\|_{L^\infty(0,T;L^\infty)}\\ &\leqslant C\sum_{j'\geqslant j-3}2^{-j'(1+2\alpha)}\|\tilde{\kappa}\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})}\|\theta\|_{L^\infty(0,T;C^\alpha)}\\ &\leqslant C2^{-j(1+2\alpha)}\|\theta\|_{L^\infty(0,T;C^\alpha)}\|\tilde{\kappa}\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})}.\end{aligned}$$

In view of the definition of $T'_{\Delta_j\nabla\theta}\tilde{\kappa}$, we find that

$$T'_{\Delta_j\nabla\theta}\tilde{\kappa}=\sum_{j'\geqslant j-2}S_{j'+2}\Delta_j\nabla\theta\Delta_{j'}\tilde{\kappa},$$

from which and Lemma 2.1, it follows that

$$\begin{aligned}\big\|T'_{\Delta_j\nabla\theta}\tilde{\kappa}\big\|_{L^2(0,T;L^2)} &\leqslant C\sum_{j'\geqslant j-2}2^j\|\Delta_j\theta\|_{L^\infty(0,T;L^\infty)}\|\Delta_{j'}\tilde{\kappa}\|_{L^2(0,T;L^2)}\\ &\leqslant C2^{j(1-\alpha)}\|\theta\|_{L^\infty(0,T;C^\alpha)}\sum_{j'\geqslant j-2}\|\Delta_{j'}\tilde{\kappa}\|_{L^2(0,T;L^2)}\\ &\leqslant C2^{-j(1+2\alpha)}\|\theta\|_{L^\infty(0,T;C^\alpha)}\|\tilde{\kappa}\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})}.\end{aligned}$$

Now, we turn to estimate $[\Delta_j,T_{\tilde{\kappa}}]\nabla\theta$. In view of the definition of Δ_j, we write

$$\begin{aligned}[\Delta_j,T_{\tilde{\kappa}}]\nabla\theta &= \sum_{|j'-j|\leqslant 4}[S_{j'-1}\tilde{\kappa},\Delta_j]\nabla\Delta_{j'}\theta\\ &=\sum_{|j'-j|\leqslant 4}2^{2j}\int_{\mathbf{R}^2}\varphi\big(2^j(x-y)\big)\big(S_{j'-1}\tilde{\kappa}(x)-S_{j'-1}\tilde{\kappa}(y)\big)\nabla\Delta_{j'}\theta(y)\,dy\end{aligned}$$

$$= \sum_{|j'-j|\leqslant 4} 2^{3j} \int_{\mathbf{R}^2} \int_0^1 y \cdot \nabla S_{j'-1}\tilde{\kappa}(x-\tau y)\,d\tau\, \nabla\varphi\big(2^j y\big)\Delta_{j'}\theta(x-y)\,dy$$
$$+ \sum_{|j'-j|\leqslant 4} 2^{2j} \int_{\mathbf{R}^2} \varphi\big(2^j(x-y)\big)\nabla S_{j'-1}\tilde{\kappa}(y)\Delta_{j'}\theta(y)\,dy,$$

from which and the Minkowski inequality, we deduce that

$$\begin{aligned}\big\|[\Delta_j, T_{\tilde{\kappa}}]\nabla\theta\big\|_{L^2(0,T;L^2)} &\leqslant C \sum_{|j'-j|\leqslant 4} \|\nabla S_{j'-1}\tilde{\kappa}\|_{L^\infty(0,T;L^\infty)} \|\Delta_{j'}\theta\|_{L^2(0,T;L^2)} \\ &\leqslant C2^{-j(1+2\alpha)} \|\tilde{\kappa}\|_{L^\infty(0,T;C^\alpha)} \|\theta\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})}.\end{aligned}$$

Summing up all the above estimates, we conclude that

$$\begin{aligned}&\big\|[\Delta_j, \kappa]\nabla\theta\big\|_{L^2(0,T;L^2)} \\ &\quad\leqslant C2^{-j(1+2\alpha)}\big(\|\tilde{\kappa}\|_{L^\infty(0,T;C^\alpha)} \|\theta\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})} + \|\theta\|_{L^\infty(0,T;C^\alpha)} \|\tilde{\kappa}\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})}\big) \\ &\quad\leqslant C2^{-j(1+2\alpha)} \|\theta\|_{L^\infty(0,T;C^\alpha)} \|\theta\|_{\widetilde{L}^2(0,T;B^{2+\alpha}_{2,\infty})},\end{aligned}$$

where C depends on $\|\theta\|_{L^\infty(0,T;L^\infty)}$. $\square$

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